

Contents

I Reversed left-BCK algebras, a framework for a unitary treatment of some commutative algebras of logic 19

1 Some commutative algebras of logic	21
1.1 Introduction	22
1.2 def. of the comm. <i>l</i> -group and surveyed algebras	25
1.2.1 Abelian <i>l</i> -groups	25
1.2.2 Boolean algebras	27
1.2.3 Heyting algebras	28
1.2.4 (Bounded) residuated lattices	29
1.2.5 Hilbert algebras	29
1.2.6 MV algebras	31
1.2.7 (Bounded) divisible residuated lattices	34
1.2.8 (Bounded) BCK algebras	35
1.2.9 Wajsberg algebras	37
1.2.10 BL algebras, Product algebras, Gödel algebras	38
1.2.11 (Weak-) R_0 algebras	40
1.2.12 MTL algebras = weak-BL algebras	40
1.2.13 IMTL, WNM, NM algebras	41
1.3 Bounded alg. versus general.-bounded alg.	41
1.4 Left-algebras versus right-algebras	44
1.4.1 T-norms, commutative monoids, pocims, pocirms, filters	44
1.4.2 (Reversed) residuums, commutative residoids, pocirs, poc-tirs, deductive systems	47
1.4.3 Left-algebras, right-algebras	50
1.4.4 Left-MV algebras	52
1.4.5 (Reversed) left-BCK algebras	54
1.5 Pocirs versus pocims, ded. systems versus filters	55
1.5.1 Reversed left-BCK algebras (i.e. left-pocirs). Some properties	55
1.5.2 Reversed left-BCK(P) algebras (i.e. left-pocirts) and reversed left-BCK(RP) algebras	58

1.5.3	Left-X-BCK(R) algebras (i.e. left-pocrims) and left-X-BCK(PR) algebras	61
1.5.4	Left-pocrims and left-pocrims are equivalent and deductive systems coincide with filters	65
1.6	The clarification of the connections	67
1.6.1	The first classification: bounded algebras, generalized-bounded algebras	68
1.6.2	The second classification: left-algebras, right-algebras	69
1.6.3	The third classification: four possible definitions of some left- algebras (right-algebras)	70
1.7	A methodology ... unifying tools	72
1.7.1	The first step: choose between bounded algebras and generalized- bounded algebras. The first unifying tool	72
1.7.2	The second step: choose between "left" and "right" algebras. The second unifying tool	72
1.7.3	The third step: choose between residuum and reversed residuum and then between four possible definitions of some left-algebras (right-algebras). The third unifying tool	73
1.8	Conclusions	79
2	An attempt to treat unitarily ... New alg.	83
2.1	(Bounded) BCK algebras. Other conditions	86
2.1.1	(Bounded) BCK algebras	86
2.1.2	(Bounded) BCK algebras. Other conditions	90
2.2	(Bounded) BCK(P) lattices. Other conditions	94
2.2.1	(Bounded) BCK(P) lattices	94
2.2.2	(Bounded) BCK(P) lattices. Other conditions	96
2.3	(Bounded) BCK(P) lattices with (prel), (div)...	100
2.3.1	(Bounded) BCK(P) lattices with (prel) or/and (div) conditions	100
2.3.2	(Bounded) BCK(P) lattices with (prel) or/and (div). Other conditions	102
2.4	Conclusions	107
2.4.1	The table "old name" - "new name" of involved algebras	108
2.4.2	The table "generalized-bounded algebra" - "bounded algebra"	108
2.4.3	The list of open problems	109
2.4.4	The conditions (DN) and (WNM): some hierarchies	110
3	New algebras. The ordinal product ...	113
3.1	... equivalent conditions with (div), (prel)	114
3.1.1	Equivalent conditions with (div)	114
3.1.2	Equivalent conditions with (prel)	116
3.1.3	Connection with condition (C)	118
3.2	... the decompositions of (div), (prel)	118
3.3	BCK(P) lattices: new algebras	122

3.4	Bounded BCK(P) lattices: new algebras	125
3.5	The ordinal product	129
3.6	Conclusions	146
3.6.1	The list of open problems from Chapter 3	147
4	Classes of examples of MV, BL and div.	149
4.1	Classes of examples of Wajsberg algebras	150
4.1.1	Classes of examples of finite linearly ordered Wajsberg algebras (MV algebras) and (WNM) Wajsberg algebras $((WNM)MV$ algebras)	160
4.1.2	Classes of examples of non-linearly ordered Wajsberg algebras (MV algebras) and (WNM) Wajsberg algebras $((WNM)MV$ algebras)	162
4.2	Classes of examples of BL algebras	168
4.2.1	Classes of examples of finite, linearly ordered, proper Hájek(P) algebras (BL algebras) and (WNM) Hájek(P) algebras $((WNM)BL$ algebras)	169
4.2.2	Classes of examples of finite, non-linearly ordered, proper Hájek(P) algebras (BL algebras) and (WNM) Hájek(P) algebras $((WNM)BL$ algebras)	173
4.2.3	Classes of examples of infinite, proper Hájek(P) algebras (BL algebras)	182
4.3	Classes of examples of bounded divisible	186
4.3.1	Classes of examples of finite, divisible bounded BCK(P) lattices and $(WNM)BCK(P)$ lattices	187
4.3.2	Classes of examples of infinite proper divisible bounded BCK(P) lattices	198
4.4	Conclusions	199
5	Classes of examples of IMTL, MTL, $\alpha\gamma$	203
5.1	Classes of examples of NM, IMTL algebras	204
5.1.1	Classes of examples of linearly ordered, proper NM and IMTL algebras	208
5.1.2	Classes of examples of non-linearly ordered, proper NM algebras and IMTL algebras	213
5.2	Classes of examples of MTL, WNM algebras	217
5.2.1	Classes of examples of linearly ordered, proper MTL and WNM algebras	217
5.2.2	Classes of examples of non-linearly ordered, proper MTL algebras and WNM algebras	225
5.3	Classes of examples of $\alpha\gamma^b$ algebras	230
5.3.1	Classes of examples of proper $(WNM)\alpha\gamma$ algebra	231
5.3.2	Classes of examples of proper $\alpha\gamma^b$ algebras	232
5.4	Conclusions	234

6 Examples of α^b, β^b, γ^b, $\beta\gamma^b$, BCK(P)-L^b	237
6.1 ... examples of $\alpha_{(DN)}$ and α^b	240
6.1.1 Classes of examples of proper $(WNM)\alpha_{(DN)}$ algebras and $\alpha_{(DN)}$ algebras	240
6.1.2 Classes of examples of proper $(WNM)\alpha$ and α^b algebras	245
6.2 ... examples of $\beta_{(DN)} = \gamma_{(DN)} = \beta\gamma_{(DN)}$, $\beta\gamma^b$, β^b , γ^b	248
6.2.1 A class of examples of proper $\beta\gamma_{(DN)} = \beta_{(DN)} = \gamma_{(DN)}$ algebra	249
6.2.2 Classes of examples of proper $(WNM)\beta\gamma$ and $\beta\gamma^b$ algebras	250
6.2.3 Classes of examples of proper $(WNM)\beta$ and β^b algebras	255
6.2.4 Classes of examples of proper $(WNM)\gamma$ and γ^b algebras	258
6.3 ... examples of BCK(P) _(DN) , BCK(P) ^b lattices	263
6.3.1 Classses of examples of proper BCK(P) _(DN) lattices (Girard monoids)	263
6.3.2 Classes of examples of proper (WNM) BCK(P) and BCK(P) ^b lattices	267
7 Examples of bounded BCK algebras	275
7.1 Examples of BCK ^b lattices, without (P)	275
7.2 Examples of BCK ^b algebras not lattices	277
7.2.1 Example of bounded BCK algebra with conditions (P) and (DN), which is not a lattice	277
7.2.2 Example of bounded BCK algebra which satisíes condition (P) and which is not a lattice	279
7.3 ... BCK ^b algebras not lattices, without (P)	280
8 Final conclusions of Part I	283
8.1 Summary of the most important results	285
8.2 Final remarks of Part I	286
8.2.1 Summary of open problems through Chapters 1-7	290
II Reversed left-pseudo-BCK algebras, a framework for a unitary treatment of some non-commutative algebras of logic	293
9 Some non-commutative algebras of logic	295
9.1 Introduction	296
9.2 def. of <i>l</i> -group and surveyed algebras	299
9.2.1 <i>l</i> -groups	299
9.2.2 (Bounded) pseudo-residuated lattices	300
9.2.3 (Bounded) divisible pseudo-residuated lattices	300
9.2.4 Pseudo-MV algebras	301
9.2.5 Pseudo-Wajsberg algebras	303

9.2.6	Pseudo-BL algebras, pseudo-Product algebras	304
9.2.7	(Bounded) pseudo-BCK algebras	306
9.2.8	Weak-pseudo-BL algebras (pseudo-MTL algebras)	306
9.2.9	Pseudo-IMTL, pseudo-WNM, pseudo-NM algebras	307
9.2.10	(Weak-) pseudo-R ₀ algebras	307
9.3	Bounded algebras versus gen.-bounded algebras	309
9.4	Left-algebras versus right-algebras	310
9.4.1	Pseudo-t-norms, monoids, poims, porims, filters	310
9.4.2	(Reversed) pseudo-residuums, residoids, poirs, potirs, deductive systems	312
9.4.3	Left-algebras, right-algebras	316
9.4.4	Left-pseudo-MV algebras	318
9.4.5	(Reversed) left-pseudo-BCK algebras	319
9.5	Poirs versus poims, ded. syst. versus filters	320
9.5.1	Reversed left-pseudo-BCK algebras (= left-poirs). Some properties	321
9.5.2	Reversed left-pseudo-BCK(pP) algebras (= left-potirs) and reversed left-pseudo-BCK(pRP) algebras	323
9.5.3	Left-X-pseudo-BCK(pR) algebras (i.e. left-porims) and left-X-pseudo-BCK(pPR) algebras	326
9.5.4	Left-potirs and left-porims are equivalent and deductive systems coincide with filters	331
9.6	The clarification of the connections	334
9.6.1	The first classification: bounded algebras, generalized-bounded algebras	335
9.6.2	The second classification: left-algebras, right-algebras	336
9.6.3	The third classification: four possible definitions of some left-algebras (right-algebras)	337
9.7	A methodology ... unifying tools	338
9.7.1	The first step: choose between bounded algebras and generalized-bounded algebras. The first unifying tool	338
9.7.2	The second step: choose between "left" and "right" algebras. The second unifying tool	339
9.7.3	The third step: choose between pseudo-residuum and reversed pseudo-residuum and then between four possible definitions of some left-algebras (right-algebras). The third unifying tool	340
9.8	Conclusions	346
10	An attempt to treat unitarily ... New alg.	349
10.1	(Bounded) pseudo-BCK algebras. Other conditions	352
10.1.1	(Bounded) pseudo-BCK algebras	352
10.1.2	(Bounded) pseudo-BCK algebras. Other conditions	357
10.2	(Bounded) pseudo-BCK(pP) lattices	366

10.2.1 (Bounded) pseudo-BCK(pP) lattices	366
10.2.2 (Bounded) pseudo-BCK(pP) lattices. Other conditions	368
10.3 (Bounded) pseudo-BCK(pP) latt. with (pprel), (pdiv)...	372
10.3.1 (Bounded) pseudo-BCK(pP) lattices with (pdiv) or/and (pprel) conditions	373
10.3.2 (Bounded) pseudo-BCK(pP) lattices with (pdiv) or/and (pprel). Other conditions	375
11 New algebras. The ordinal product ...	381
11.1 ... equivalent conditions with (pdiv), (pprel)	382
11.1.1 Equivalent conditions with (pdiv)	383
11.1.2 Equivalent conditions with (pprel)	386
11.1.3 Connection with (pC)	389
11.2 ... the decompositions of (pdiv), (pprel)	389
11.3 Pseudo-BCK(pP) lattices: new algebras	395
11.4 Bounded pseudo-BCK(pP) lattices: new algebras	399
11.4.1 The four planes \mathcal{P}^b , $(WNM)\mathcal{P}$, $\mathcal{P}_{(pDN)}$ and $(WNM)\mathcal{P}_{(pDN)}$	403
11.4.2 The four planes \mathcal{P}^b , $(WNM)\mathcal{P}$, $\mathcal{P}_{(good)}$ and $(WNM)\mathcal{P}_{(good)}$	407
11.5 The ordinal product	410
11.6 Conclusions	423
11.6.1 The list of open problems from Chapter 11	424
12 Examples of pseudo-MV, pseudo-BL and div. ...	427
12.1 Classes of examples of pseudo-MV algebras	427
12.1.1 Classes of examples of linearly ordered pseudo-Wajsberg al- gebras (pseudo-MV algebras)	430
12.1.2 Classes of examples of non-linearly ordered left-pseudo-Wajsberg algebras (left-pseudo-MV algebras)	440
12.2 Classes of examples of pseudo-BL algebras	443
12.2.1 Classes of examples of linearly ordered pseudo-BL algebras	445
12.2.2 Examples of non-linearly ordered pseudo-BL algebras	448
12.3 Classes of ex. of div. b. pseudo-res. lattices	454
12.4 Conclusions	461
13 Ex. of pseudo-NM, pseudo-MTL, pseudo-$\alpha\gamma^b$...	463
13.1 Classes of examples of pseudo-NM algebras	463
13.1.1 Classes of examples of linearly ordered pseudo-R ₀ algebras (pseudo-NM algebras	471
13.1.2 Classes of examples of non-linearly ordered left-pseudo-R ₀ algebras (left-pseudo-NM algebras)	480
13.2 Classes of examples of pseudo-MTL algebras	482
13.2.1 Classes of examples of linearly ordered pseudo-MTL algebras (weak-pseudo-Hájek(pP) algebra)	483

13.2.2 Classes of examples of non-linearly ordered pseudo-MTL algebras	487
13.2.3 Example of finite proper $(pWNM)p\alpha\beta$ algebras (pseudo-WNM algebras) and of proper $(pWNM)p\alpha\beta_{good}$ algebra (good pseudo-WNM algebra)	491
13.2.4 Examples of finite proper $p\alpha\beta_{(good)}$ algebras (good pseudo-MTL algebras) and of proper $p\alpha\beta^b$ algebras (pseudo-MTL algebras)	494
13.3 Classes of examples of bounded pseudo- $\alpha\gamma$ algebras	497
13.3.1 Classes of examples of the Forms 1'', 2'', 3''	497
13.3.2 Examples of proper finite pseudo- $\alpha\gamma^b$ algebras and $(pWNM)$ pseudo- $\alpha\gamma$ algebras	501
13.4 Conclusions	505
14 Ex. of pseudo-α^b and b. pseudo-res. lattices	507
14.1 Example of finite proper bounded pseudo- α algebra	507
14.2 Example of bounded pseudo-BCK(pP) lattice	509
14.2.1 Example of proper bounded pseudo-BCK(pP) lattice	509
14.2.2 Example of proper good pseudo-BCK(pP) lattice	511
15 Final conclusions of Part II	515
15.1 Summary of the most important results	517
15.2 Final remarks of Part II	518
15.2.1 Summary of open problems through Chapters 9-14	521